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The generalized Lagrangian-mean equations and hydrodynamic stability

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The generalized Lagrangian-mean (GLM) formulation of Andrews & McIntyre (1978*a*, *b*) offers alternative physical concepts and possible saving of effort in calculation, as compared with the more conventional Eulerian-mean approach. Though most existing applications of this theory concern waves on weakly sheared mean flows, it is also suitable for study of waves in strong shear flows. The hydrodynamic stability of parallel shear flows is examined from this point of view. An appreciation is gained of the roles of Stokes drift, pseudomomentum, energy and pseudoenergy in this context, such understanding being a necessary prerequisite for future developments. Several known results of linear stability theory, including the inflexion-point and semicircle theorems, are concisely rederived from the GLM conservation laws.

1. Introduction

There is a wide class of fluid-flow phenomena that exhibit motions having both a mean and a fluctuating part. Typically, weakly nonlinear oscillations tend to induce weak mean flows, and slowly varying mean flows may produce slow changes in wave properties. Also, strong mean shear flows are often linearly unstable to wavelike disturbances, which may grow and modify the mean flow. The understanding of nonlinear processes that couple mean and fluctuating motions presents a considerable challenge. Until recently, such problems have mostly been tackled *via* the Eulerian equations of mean motion; but Andrews & McIntyre (1978*a*, *b*, hereinafter referred to as I and II) have developed, in very general form, an alternative approach based upon Lagrangian rather than Eulerian averages. Their 'generalized Lagrangian-mean' (GLM) equations are, in fact, a hybrid Eulerian–Lagrangian description in which Lagrangian-mean flow quantities satisfy equations in Eulerian form with position $\mathbf{x}$ and time t as independent variables (i.e. $\mathbf{x}$ replaces the usual Lagrangian 'labels' designating initial particle locations).

The GLM formulation has helped clarify wave-related concepts such as 'wave action' and 'pseudomomentum' and may often provide a more-direct method of calculation that the Eulerian approach. For instance, Leibovich (1980) has rederived the so-called 'Craik–Leibovich equations' for Langmuir-vortex flows below the ocean surface much more briefly than in the original Eulerian version (Craik & Leibovich 1976; Leibovich 1977). In essence, the Eulerian equations are less well-suited for handling flow quantities that follow individual fluid particles, and so tend to conceal the simple role played by the Stokes drift in vortex-line deformation. Other examples illustrating the use of the GLM equations are given by McIntyre (1980) and Grimshaw (1979, 1982).

In the present paper we outline the basic GLM theory for growing waves in incompressible parallel shear flows, in the belief that this formulation may find useful application. The following paper (Craik 1982) employs this formulation to investigate the stability of two-dimensional wavelike flows, with mean shear, to disturbances of 'longitudinal-vortex' form.

2. Particle paths and averages

We first consider a two-dimensional Eulerian velocity field $\mathbf{u} = [u, 0, w]$ with space coordinates (x, y, z) and time t. For simplicity, attention is restricted to a constant-density fluid, for which continuity requires that $\mathbf{u}$ be solenoidal. Supposing this field to comprise a primary unidirectional shear flow $[\bar{u}(z), 0, 0]$ and a small $O(\epsilon)$ wavelike disturbance along with associated terms of higher order in ϵ, we write

$$u(x,z,t) = \bar{u}(z) + \mathrm{Re}\{\epsilon\phi'(z)\, e^{i\alpha x+\sigma t}\} + O(\epsilon^2), \tag{2.1a}$$

$$w(x,z,t) = \mathrm{Re}\{-\epsilon i\alpha\phi(z)\, e^{i\alpha x+\sigma t}\} + O(\epsilon^2). \tag{2.1b}$$

Here $\sigma \equiv \alpha c_i$ denotes the temporal growth or decay rate of the wave, α is its real wavenumber, and the reference frame has been chosen to move in the direction of the primary flow with the phase speed of the wave. Note that $\bar{u}(z) = U(z) - c_r$, where $U(z)$ is the velocity profile and c_r the observed phase speed in some parallel reference frame. When $\sigma = 0$ the wave is neutrally stable and the wave profile is at rest in the chosen frame. The prime denotes d/dz and $\phi(z)$ is the appropriate eigenfunction of the linearized disturbance equation, usually the Orr–Sommerfeld equation, and suitable boundary conditions. For rigid plane boundaries at $z = z_1$ and z_2,

$$\phi = \phi' = 0 \quad (z = z_1, z_2).$$

If viscosity is neglected, the Orr–Sommerfeld equation is replaced by Rayleigh's equation

$$(\bar{u} - ic_i)(\phi'' - \alpha^2\phi) - \bar{u}''\phi = 0, \tag{2.2}$$

and the reduced boundary conditions are

$$\phi = 0 \quad (z = z_1, z_2). \tag{2.3}$$

A fluid particle situated at $\mathbf{x} = (X_0, Y_0, Z_0)$ at some initial time $t = t_0$ has position coordinates (X, Y, Z) at later times $t > t_0$ given by

$$X(t) = X_0 + \int_{t_0}^{t} u[X(s), Z(s), s]\, ds, \tag{2.4a}$$

$$Y(t) = Y_0, \tag{2.4b}$$

$$Z(t) = Z_0 + \int_{t_0}^{t} w[X(s), Z(s), s]\, ds, \tag{2.4c}$$

where $u(x,z,t), w(x,z,t)$ are as given above. With just the primary flow,

$$X(t) = X_0 + \bar{u}(Z_0)(t - t_0), \quad Z(t) = Z_0;$$

while, to the next order of approximation,

$$Z(t) = Z_0 + \epsilon\, \mathrm{Re} \int_{t_0}^{t} -i\alpha\phi(Z_0)\, e^{i\alpha[X_0 + \bar{u}(Z_0)s]}\, e^{\sigma s}\, ds + O(\epsilon^2)$$

$$= Z_0 + \epsilon\, \mathrm{Re}\left\{\left(\frac{\phi(Z_0)\, e^{i\alpha X_0}}{\bar{u}(Z_0) - ic_i}\right)(e^{\sigma t_0} - e^{\sigma t}\, e^{i\alpha\bar{u}(Z_0)t})\right\} + O(\epsilon^2).$$

If we consider a set of particles initially distributed at fixed Y_0 and Z_0 but at *all* values of X_0 (with uniform number density), it is clear that the average position Z, taken with respect to the label X_0, is $Z_0+O(\epsilon^2)$. On the other hand, the average position Z of a single particle, taken with respect to time t with fixed X_0, has an $O(\epsilon)$ contribution. In the particular case $\sigma = 0$, this average is

$$Z_0+\epsilon\,\mathrm{Re}\left\{\frac{\phi(Z_0)\,e^{i\alpha X_0}}{\bar{u}(Z_0)}\right\}+O(\epsilon^2).$$

This result reflects the fact that particles initially located at differing values of X_0, but the same level Z_0, do not normally lie on the same streamline. The mean levels of the streamlines that carry such particles lie in a band of width $\epsilon|\phi(Z_0)/\bar{u}(Z_0)|$ centred on Z_0.

On proceeding to order ϵ^2, one finds that averages of $Z(t)-Z_0$ and $X(t)-X_0-\bar{u}(Z_0)\,(t-t_0)$, taken with respect to X_0 for fixed Z_0, are normally time-dependent even when $\sigma = 0$. When $\sigma = 0$, oscillations occur with frequency $\alpha\bar{u}(Z_0)$, which is just the frequency with which individual particles pass successive wave crests. Such periodicity normally occurs when different particles of an averaged ensemble are located on different streamlines, or when particles on the same streamline are unevenly distributed between peaks and troughs.

A non-zero mean of $Z-Z_0$ at order ϵ^2, taken relative to X_0, indicates that the average level of a set of particles originally at Z_0 differs from Z_0. Similarly, the mean level of an individual particle initially situated at (X_0, Y_0, Z_0) typically differs from Z_0. These results are of importance in interpreting the Lagrangian-mean quantities of I. For instance, the Lagrangian-mean $\bar{u}^{\mathrm{L}}(z)$ may be identified with the velocity of the centre of mass of a row of fluid particles that initially extended in a straight line along the x-direction; but the original level of this column was generally not at z but at some $z' = z+O(\epsilon^2)$.

In fact, the formation of successive approximations for X and Z in powers of ϵ, by repeated substitution in the right-hand side of (2.4), leads to unnecessary complications when the particles to be averaged start from the same initial level Z_0 in the presence of a wave. The best way of circumventing these for growing disturbances ($\sigma > 0$) is to choose $t_0 = -\infty$; the particles to be averaged then being initially spaced evenly along $Z = Z_0$ when the wave amplitude is effectively zero. This provides an initialization satisfying postulate (viii) of I while allowing the disturbance to be treated as free of external forcing, a situation that permits of simplification in GLM theory. For neutral waves ($\sigma = 0$), special difficulties are encountered in the vicinity of critical layers where $\bar{u}(z) = 0$ (see I, §10). In such regions, the streamlines are closed 'cat's eyes' while the averaging procedure gives simple results only for 'open' streamlines. But such difficulties are here circumvented by considering the limit $\sigma \to 0$ for growing disturbances.

3. Stokes drift and pseudomomentum

The Eulerian mean velocity $\bar{\mathbf{u}}^{\mathrm{E}}$ and Lagrangian mean velocity $\bar{\mathbf{u}}^{\mathrm{L}}$ are related by

$$\bar{\mathbf{u}}^{\mathrm{L}} = \bar{\mathbf{u}}^{\mathrm{E}}+\bar{\mathbf{u}}^{\mathrm{S}},$$

where $\bar{\mathbf{u}}^{\mathrm{S}}$ is the generalized 'Stokes drift'. The difference between $\bar{\mathbf{u}}^{\mathrm{E}}$ and $\bar{\mathbf{u}}^{\mathrm{L}}$ was first recognized by Stokes (1847) in the context of water-wave theory. Andrews &

McIntyre's generalization defines $\bar{\mathbf{u}}^S$ in tensor form as

$$\bar{u}_i^S = \overline{\xi_j \hat{u}_{i,j}} + \tfrac{1}{2}\overline{\xi_j \xi_k}\,\bar{u}_{i,jk} + O(\epsilon^3) \tag{3.1}$$

(see I, equation (2.27)) for small-amplitude waves of order $O(\epsilon)$. Here, $\hat{u}_i$ denotes the $O(\epsilon)$ Eulerian velocity fluctuations and ξ_i represents the displacements of fluid particles from their mean positions. That is, ξ_i and $\hat{u}_i$ are quantities with zero mean. Commas denote partial differentiation. We shall regard the overbar as denoting x-averages in the sense described at the end of the preceding section, but other choices of averaging procedure are possible (see I, §2). The Eulerian mean $\bar{u}_1^E$ differs from the primary flow $\bar{u}$ by an $O(\epsilon^2)$ quantity, which will be denoted by $\bar{\bar{u}}_1$.

In the GLM formulation, a central role is played by the pseudomomentum per unit mass $\mathbf{p} \equiv p_i$ $(i = 1, 2, 3)$, defined as

$$p_i = -\overline{\xi_{j,i}\, u_j^\ell}$$

in non-rotating reference frames. Here u_i^ℓ denotes the fluctuating part of the Lagrangian velocity field, which is

$$u_i^\ell = \hat{u}_i + \xi_j \bar{u}_{i,j} + O(\epsilon^2)$$

for small-amplitude waves. For two-dimensional flows of the form (2.1), the $O(\epsilon)$ Eulerian velocity fluctuations $\hat{u}_i$ and particle displacements ξ_i are

$$\left.\begin{aligned} \hat{u}_1 &= \epsilon\,\mathrm{Re}\,\{\phi'(z)\, e^{i\alpha x + \sigma t}\}, \quad \hat{u}_2 = 0, \quad \hat{u}_3 = \epsilon\,\mathrm{Re}\,\{-i\alpha\phi(z)\, e^{i\alpha x+\sigma t}\}, \\ \xi_1 &= \epsilon\,\mathrm{Re}\left\{\left(\frac{\phi}{\bar{u}-ic_i}\right)'\frac{e^{i\alpha x+\sigma t}}{i\alpha}\right\}, \quad \xi_2 = 0, \quad \xi_3 = \epsilon\,\mathrm{Re}\left\{\frac{-\phi\, e^{i\alpha x+\sigma t}}{\bar{u}-ic_i}\right\}, \end{aligned}\right\} \tag{3.2}$$

where $\sigma = \alpha c_i$. In view of (2.2), the z-displacement ξ_3 satisfies the adjoint Rayleigh equation

$$(\bar{u}-ic_i)(\xi_3'' - \alpha^2\xi_3) + 2\bar{u}'\xi_3' = 0.$$

The components of Stokes drift $\bar{u}_i^S$ and pseudomomentum p_i are, at $O(\epsilon^2)$,

$$\bar{u}_1^S = \tfrac{1}{4}\epsilon^2 e^{2\sigma t}\left[-\left(\frac{\phi\phi^{*\prime}}{\bar{u}-ic_i} + \text{c.c.}\right)' + \frac{\bar{u}''|\phi|^2}{\bar{u}^2 + c_i^2}\right], \tag{3.3a}$$

$$\bar{u}_2^S = 0, \quad \bar{u}_3^S = \tfrac{1}{2}\epsilon^2 e^{2\sigma t}\alpha c_i\left(\frac{|\phi|^2}{\bar{u}^2+c_i^2}\right)', \tag{3.3b, c}$$

$$p_1 = -\tfrac{1}{2}\epsilon^2 e^{2\sigma t}\bar{u}\left\{\left|\left(\frac{\phi}{\bar{u}-ic_i}\right)'\right|^2 + \alpha^2\left|\frac{\phi}{\bar{u}-ic_i}\right|^2\right\} = -\alpha^2\bar{u}\overline{|\xi|^2}, \tag{3.4a}$$

$$p_2 = 0, \quad p_3 = -\tfrac{1}{4}\epsilon^2 e^{2\sigma t}\left\{\left(\frac{\phi^*}{\bar{u}+ic_i}\right)'\left[\left(\frac{\bar{u}+ic_i}{i\alpha}\right)\left(\frac{\phi}{\bar{u}-ic_i}\right)'' + i\alpha\phi\right] + \text{c.c.}\right\}, \tag{3.4b, c}$$

where ϕ^* is the complex conjugate of ϕ, and c.c. denotes the complex conjugate of the term it follows.

For neutral waves, (3.3c) gives $\bar{u}_3^S = 0$, at least outside the critical layer. Also, $p_3 = 0$ for neutral waves governed by the *inviscid* equation (2.2); but it is apparently non-zero for neutral waves governed by the viscous Orr–Sommerfeld equation.

With the plane-wall boundary conditions (2.3),

$$\int_{z_1}^{z_2} \bar{u}_1^S\, dz = \tfrac{1}{4}\epsilon^2 e^{2\sigma t}\int_{z_1}^{z_2}\frac{\bar{u}''|\phi|^2}{\bar{u}^2+c_i^2}\,dz, \tag{3.5}$$

which is zero for inviscid unstratified flows.

For the present averaging procedure, the pseudomomentum p_1 equals Andrews & McIntyre's 'generalized wave action' A (see II, §2: for other averages, p_1 and A differ only by a multiplicative constant) and the general wave-action equation II (2.15) yields

$$\rho \frac{\partial p_1}{\partial t} \equiv \rho \frac{\partial A}{\partial t} = -\nabla . \mathbf{B} = -\frac{\partial \overline{(\hat{p} \partial \xi_3 / \partial x)}}{\partial z}$$

at $O(\epsilon^2)$ for *inviscid* flows, where $\hat{p}$ is the $O(\epsilon)$ Eulerian pressure fluctuation. It follows that

$$\int_{z_1}^{z_2} p_1 \, dz = \text{constant},$$

and the constant is zero since $p_1 = 0$ at $t = -\infty$. This conservation of pseudomomentum therefore yields

$$\int_{z_1}^{z_2} \bar{u} |\xi|^2 \, dz = 0. \tag{3.6}$$

The well-known result that the wave velocity c_r lies within the range of the flow velocity immediately follows.

The x-component of the GLM equations (I (5.5a)) is

$$\bar{D}^{\mathrm{L}}(\bar{u}_1^{\mathrm{L}} - p_1) \equiv \left(\frac{\partial}{\partial t} + \bar{\mathbf{u}}^{\mathrm{L}} . \nabla\right)(\bar{u}_1^{\mathrm{L}} - p_1) = 0,$$

in the absence of viscosity and any mean pressure gradient. At $O(\epsilon^2)$, this is

$$\frac{\partial(\bar{\bar{u}}_1 + \bar{u}_1^{\mathrm{S}} - p_1)}{\partial t} + \bar{u}_3^{\mathrm{S}} \bar{u}' = 0,$$

where $\bar{\bar{u}}_1$ is the $O(\epsilon^2)$ Eulerian-mean velocity. Substitution of (3.3a, c) and (3.4a) leads immediately to

$$\bar{\bar{u}}_1 = \tfrac{1}{4}\epsilon^2 \, e^{2\sigma t} \left(\frac{\bar{u}'' |\phi|^2}{\bar{u}^2 + c_i^2}\right) = \tfrac{1}{2} \bar{u}'' \overline{\xi_3^2}, \tag{3.7}$$

a result customarily derived by integrating the Eulerian equation

$$\frac{\partial \bar{\bar{u}}_1}{\partial t} = -\frac{\partial \overline{(\hat{u}_1 \hat{u}_3)}}{\partial z}.$$

The net Eulerian volume flux satisfies

$$\int_{z_1}^{z_2} \bar{\bar{u}}_1 \, dz = 0, \tag{3.8}$$

in accordance with conservation of x-momentum. It follows that $\bar{u}''$ must change sign within the flow domain $[z_1, z_2]$ if amplified waves exist.

4. Pseudoenergy and the semicircle theorem

For inviscid flows, a 'pseudoenergy–pseudomomentum tensor' $T_{\mu\nu}$, where μ and ν stand for x_i ($i = 1, 2, 3$) or t, satisfies the conservation relations

$$T_{\mu\nu,\nu} = 0$$

for each μ (see II, §5). The component T_{tt} is the 'pseudoenergy' (per unit volume) and the components T_{tj} ($j = 1, 2, 3$) denote its flux. In the present case

$$T_{tt,t} + T_{t3,3} = 0,$$

and so the net pseudoenergy is

$$\int_{z_1}^{z_2} T_{tt}\,dz = \text{constant}, \tag{4.1}$$

since T_{t3} vanishes on the plane boundaries $z = z_1$ and z_2. Since T_{tt} is zero at $t = -\infty$, net pseudoenergy is conserved, remaining identically zero for all t.

From II (5.13a),

$$T_{tt} = \tilde{\rho}\mathsf{e} - \overline{L - L_0}, \tag{4.2}$$

where

$$\mathsf{e} \equiv \overline{\xi_{i,t}\hat{u}_i} + \bar{u}'\overline{\xi_{1,t}\xi_3} = \overline{\xi_{j,t}u_j^{\ell}}$$

$$\overline{L - L_0} = \tilde{\rho}[\tfrac{1}{2}\overline{(\bar{\mathbf{u}}^{\mathrm{L}} + \bar{D}^{\mathrm{L}}\boldsymbol{\xi})^2} - \tfrac{1}{2}(\bar{\mathbf{u}}^{\mathrm{L}})^2].$$

Here

$$L = \tfrac{1}{2}\tilde{\rho}(\bar{\mathbf{u}}^{\mathrm{L}} + \bar{D}^{\mathrm{L}}\boldsymbol{\xi})^2 + \text{constant},$$

is the Lagrangian per unit volume (cf. II (5.5)) and L_0 is the 'mean-field' Lagrangian obtained by neglecting from L all terms explicitly containing $\boldsymbol{\xi}$. The reference density $\tilde{\rho}$ (cf. I (4.3) and I (9.3)) is

$$\tilde{\rho} = \rho[1 + \overline{\xi_{1,1}\xi_{3,3} - \xi_{1,3}\xi_{3,1}}]$$

$$= \rho[1 - \tfrac{1}{2}\overline{(\xi_j\xi_k)}_{,jk} + O(\epsilon^3)] \tag{4.3}$$

for the two-dimensional constant-density flows considered here. The $O(\epsilon^2)$ apparent density change $\tilde{\rho} - \rho$ is related to the divergence effect noted in I that the Lagrangian-mean velocity is generally non-solenoidal.

At $O(\epsilon^2)$, the pseudoenergy is found to be

$$T_{tt} = \tfrac{1}{4}\rho(c_i^2 - \bar{u}^2)\,\alpha^2\overline{|\xi|^2},$$

and this must integrate to zero, by (4.1): accordingly, conservation of pseudoenergy requires that

$$\int_{z_1}^{z_2} (\bar{u}^2 - c_i^2)\,\overline{|\xi|^2}\,dz = 0. \tag{4.4}$$

This result may also be derived via the GLM 'virial theorem' (II (4.2)), which yields the inviscid result

$$\tfrac{1}{2}\frac{\partial^2\overline{|\xi|^2}}{\partial t^2} - \overline{|\mathbf{u}^{\ell}|^2} = -\rho^{-1}\overline{\xi_i p_{,i}}. \tag{4.5}$$

Since, from (3.2),

$$\overline{|\mathbf{u}^{\ell}|^2} = \alpha^2(\bar{u}^2 + c_i^2)\overline{|\xi|^2}$$

and $\xi_{i,i} = O(\epsilon^2)$, integration of (4.5) leads directly to (4.4). In a similar way, the virial theorem of Eckart (1963) yields both (4.4) and (3.6) as real and imaginary parts.

From these two conservation laws, (4.4) and (3.6), for pseudoenergy and pseudomomentum, Howard's (1961) semicircle theorem is readily derived (cf. Eckart 1963). Since $|\xi|^2$ is non-negative, $\bar{u}^2 - c_i^2 + \lambda\bar{u}$ must change sign somewhere within the flow domain $[z_1, z_2]$ for all constants λ. But $\bar{u} = U - c_r$ in any parallel reference frame, with c_r the observed phase speed, and we may write

$$U(z) = \tfrac{1}{2}(U_{\max} + U_{\min}) + v(z),$$

where $U_{\max}$ and $U_{\min}$ are the respective maximum and minimum flow velocities. The choice $\lambda = 2c_r - U_{\max} - U_{\min}$ leads immediately to the result that

$$v^2 - [c_r - \tfrac{1}{2}(U_{\max} + U_{\min})]^2 - c_i^2$$

must change sign in $[z_1, z_2]$. On setting v^2 equal to its greatest-possible value, it follows that

$$[c_r - \tfrac{1}{2}(U_{\max} + U_{\min})]^2 + c_i^2 < [\tfrac{1}{2}(U_{\max} - U_{\min})]^2,$$

and the semicircle theorem is proved. A similar proof may be constructed for compressible and density-stratified flows (Eckart 1963).

5. Mass flux and energy

Since the net Eulerian and Lagrangian mean mass fluxes must be equal,

$$\rho \int_{z_1}^{z_2} (\bar{u}+\bar{\bar{u}}_1)\, dz = \int_{z_1}^{z_2} \tilde{\rho}(\bar{u}+\bar{\bar{u}}_1+\bar{u}_1^{\mathrm{S}})\, dz,$$

and so, at $O(\epsilon^2)$,

$$\int_{z_1}^{z_2} [(\tilde{\rho}-\rho)\,\bar{u}+\rho\bar{u}_1^{\mathrm{S}}]\, dz = 0.$$

On using (3.5), it follows that, at $O(\epsilon^2)$,

$$\int_{z_1}^{z_2} (\tilde{\rho}-\rho)\,\bar{u}\, dz = 0 \tag{5.1}$$

for inviscid unstratified flows. Also, from (4.3),

$$\int_{z_1}^{z_2} (\tilde{\rho}-\rho)\, dz = -\int_{z_1}^{z_2} \overline{(\xi_1 \xi_{3,1})}_{,3}\, dz = 0 \tag{5.2}$$

at $O(\epsilon^2)$, in accordance with conservation of mass.

Since there is no potential energy, the Lagrangian L may be regarded as the energy per unit volume of a fluid particle. The total mean change in L due to the disturbance is $L-L_1$ where $L_1 \equiv \frac{1}{2}\rho\bar{u}^2$. Accordingly,

$$\overline{L-L_1} = \rho\bar{u}(\bar{u}_1^{\mathrm{S}}+\bar{\bar{u}}_1)+\tfrac{1}{2}(\tilde{\rho}-\rho)\,\bar{u}^2+\tfrac{1}{2}\rho\overline{(\bar{D}^{\mathrm{L}}\xi)^2}+O(\epsilon^4),$$

$$\overline{L-L_0} = \tfrac{1}{2}\rho\overline{(\bar{D}^{\mathrm{L}}\xi)^2}+O(\epsilon^4).$$

Since total energy is conserved,

$$\int_{z_1}^{z_2} \overline{L-L_1}\, dz = 0. \tag{5.3}$$

Since the total Eulerian-mean and GLM energies must be equal, the Eulerian-mean energy is also conserved. Therefore

$$E+\rho\int_{z_1}^{z_2} \bar{\bar{u}}_1 \bar{u}\, dz = 0,$$

where

$$E \equiv \tfrac{1}{4}\rho\epsilon^2\, e^{2\sigma t}\int_{z_1}^{z_2} (|\phi'|^2+\alpha^2|\phi|^2)\, dz$$

is the mean kinetic energy per unit span associated with the Eulerian fluctuations, and the second term denotes the kinetic-energy change of the mean flow.

In any parallel reference frame, with $\bar{u} = U-c_{\mathrm{r}}$, the corresponding total Eulerian-mean energy is

$$E+\rho\int_{z_1}^{z_2} \bar{\bar{u}}_1\, U\, dz = \rho c_{\mathrm{r}}\int_{z_1}^{z_2} \bar{\bar{u}}_1\, dz,$$

which is likewise zero by virtue of (3.8). That is to say, spontaneously growing disturbances in inviscid flows have zero total Eulerian-mean energy and pseudoenergy, whatever the reference frame.

In contrast, disturbances initiated by application of external forces may have positive or negative energy (cf. Cairns 1979; Craik & Adam 1979). For stratified flows and those with a free surface, there are neutrally stable modes that can only be created in this way: their pseudoenergy and energy is non-zero. Consideration of pseudoenergy, rather than energy, in such cases is likely to be advantageous, for T_{tt} may be evaluated without knowledge of the $O(\epsilon^2)$ Eulerian-mean velocity $\overline{\overline{u}}_1$. In particular, the Eulerian formulation encounters difficulties in dealing with stratified shear flows (absent for the piecewise-constant profiles of $\bar{u}$ and ρ considered by Cairns and by Craik & Adam). No great difficulties appear in the GLM approach unless for neutral modes with critical layers where $\bar{u} = 0$.

6. Pseudoenergy flux in viscous flows

The tensor component T_{t3} is

$$T_{t3} = \bar{u}_3^L \tilde{\rho} \mathsf{e} + \overline{p^\xi \xi_{m,t} K_{m3}}$$

in the notation of II, equation (5.13*b*). This equals the flux (in the x_3 direction) of pseudoenergy. For neutral waves, the displacements $\boldsymbol{\xi}$ are independent of time in the present reference frame and both T_{t3} and e are then zero.

In any parallel reference frame, $\bar{u} = U - c_r$. Since $\bar{u}_3^L \tilde{\rho} \mathsf{e}$ is $O(\epsilon^4)$ at most,

$$T_{t3} = \overline{\hat{p}\xi_{3,t}}$$

at $O(\epsilon^2)$, where $\hat{p} = \epsilon\rho\,\mathrm{Re}\{[(c-U)\,\phi' + U'\phi]\,e^{i\alpha(x-ct)}\}$

is the $O(\epsilon)$ Eulerian pressure fluctuation and $c \equiv c_r + ic_i$ is the complex wave velocity in the chosen frame. Since

$$\xi_{3,t} = \epsilon\,\mathrm{Re}\,\{i\alpha c(U-c)^{-1}\phi\, e^{i\alpha(x-ct)}\},$$

$$T_{t3} = \tfrac{1}{4}\epsilon^2 i\alpha c_r \rho(\phi'\phi^* - \phi^{*\prime}\phi) \tag{6.1}$$

for neutral waves. This is just $-c_r\tau$ where $\tau \equiv -\rho\overline{\hat{u}_1\hat{u}_3}$ is the Reynolds stress. For wholly inviscid flows, T_{t3} is zero everywhere for neutral waves.

In flows with small viscosity, leading-order viscous effects are confined to thin layers near the walls and close to critical layers. In the intervening inviscid regions, the expression (6.1) remains valid. For neutral waves, this represents a constant flux of pseudoenergy from critical layer to viscous wall layer, where energy is dissipated and pseudoenergy is dissipated or released. On the other hand, in the rest frame of a neutral wave, T_{t3} is identically zero, since the phase velocity is zero. In this frame, all the work done to sustain the disturbance against viscous dissipation derives from an $O(\epsilon^2)$ wall shear stress, provided that there is no $O(\epsilon^2)$ mean pressure gradient. The rate-of-working per unit wall area is then $-\mu\overline{\overline{u}}_{1,3}$ times the (apparent) wall velocity.

7. Discussion

The GLM theory provides alternative physical insights, which complement the more familiar Eulerian formulation of the theory of hydrodynamic stability. The meanings of the wave-related concepts of Stokes drift, pseudomomentum and pseudoenergy have been illustrated for homogeneous flows, and various results established. In particular, some known results of linear stability theory, including the semicircle and inflexion-point theorems, were concisely rederived from the GLM equations and conservation laws.

Since the GLM equations succinctly express convective processes that tend to be obscured in the Eulerian form, considerable savings of effort may sometimes accrue from using the GLM approach. This paper has described the necessary framework for cases of strong mean shear flow. A particular nonlinear application is given in the following paper (Craik 1982), which considers a novel type of instability in the form of spanwise-periodic longitudinal vortices. In that paper, a further appreciation is gained of the power and limitations of the GLM formulation.

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